The threshold-two contact process on a random r-regular graph has a first order phase transition

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Random graph G_n

Construct random graph G_n on the vertex set V_n as follows.

- Assign r "half edges" to each of the vertices.
- Pair the half edges at random (rn must be even).
- Let $\mathbb{P}$ denote the distribution of G_n.
- Condition on the event E_n that the graph is simple (no self loops or multiple edges).
- $\mathbb{P}(E_n)$ is bounded away from 0, and hence for large enough n,

$$\text{if } \tilde{\mathbb{P}} := \mathbb{P}(\cdot|E_n), \text{ then } \tilde{\mathbb{P}}(\cdot) \le c\mathbb{P}(\cdot) \text{ for some constant } c = c(r) > 0.$$

- Choose G_n according to the distribution $\tilde{\mathbb{P}}$ on simple graphs.

Threshold-two contact process

- $x \sim y$ means that x is a neighbor of y, and

$$\mathcal{N}_y := \{x \in V_n : x \sim y\}.$$

- The state of the process at time $t = 0, 1, \ldots$ is $\xi_t \subseteq V_n$, the set of occupied vertices.
- The distribution $P_{G_n,p}$ of ξ_t with parameter p conditioned on G_n is given by

$$P_{G_n,p}(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| \geq 2) = p \text{ and}$$
$$P_{G_n,p}(x \in \xi_{t+1} \mid |\mathcal{N}_x \cap \xi_t| < 2) = 0,$$

  where the decisions for different vertices at time $t+1$ are taken independently.
- Let $\mathbf{P}_p(\cdot) := \tilde{\mathbb{E}} P_{G_n,p}(\cdot)$ be the unconditional distribution of ξ_t.

The critical value p_c

- Let ξ_t^1 start from all-one configuration.
- Define the boundary p_c between convergence to the all-zero configuration within time $C(p)\log n$, and exponentially prolonged persistence as

$$p_c := \inf \left\{ p \in [0,1] : \lim_{n\to\infty} \mathbf{P}_p \left(\inf_{t \le \exp(k(p)n)} \frac{|\xi_t^1|}{n} > u(p) \right) = 1 \text{ for some } k$$

Theorem

If $r \ge 4$ and $\eta \in (0, 1/4)$, then there is an $\epsilon_1 = \epsilon_1(\eta) \in (0,1)$ such that for p sufficiently close to 1, and for some positive constants C_1 and $c_1(\eta, p)$,

$$\mathbf{P}_p \left(\inf_{t \le \exp(c_1(\eta,p)n)} \frac{|\xi_t^1|}{n} < 1 - \epsilon_1 \right) \le C_1 \exp(-c_1(\eta,p)n).$$

Remarks: So $p_c < 1$.

Behavior of ξ_t^A starting from a small occupied set A

Let ξ_t^A starts from $\xi_0^A = A$.

Theorem

There is a decreasing continuous function $\epsilon_2 : (0,1) \mapsto (0,1)$ *and a collection* $\mathcal{G}$ *of simple* r*-regular graphs on* n *vertices such that for any* $p \in (0,1)$, $C_0(p) := 2/\log(2/(1+p))$, *and any subset* $A \subset V_n$ *with* $|A| \le \epsilon_2(p) n$,

- $\sup_{G_n \in \mathcal{G}} P_{G_n,p}\left(\xi^A_{\lceil C_0(p)\log n\rceil} \neq \emptyset\right) = o(1)$,
- $\tilde{\mathbb{P}}(\mathcal{G}^c) = o(1)$.

Hence $\lim_{n\to\infty} \mathbf{P}_p\left(\xi^A_{\lceil C_0(p)\log n\rceil} \neq \emptyset\right) = 0$.

Remarks: (i) $p_c \in (0,1)$.
(ii) The density of occupied vertices does not stay in $(0, \epsilon_2)$ for long time.

Quasi-stationary density

- For $p > p_c$, let

$$\rho_n := \frac{1}{n}\left|\xi^1_{\lceil \exp(n^{1/2})\rceil}\right|.$$

  ρ_n is the quasi-stationary density of occupied vertices.
- From previous theorem, $\mathbf{P}_p(\rho_n \geq \epsilon_2(p)) = 1 - o(1)$.
- Also using monotonicity, $P_{G_n,p}, p \in [0,1]$ are stochastically ordered, i.e.

$$\text{for } p_1 < p_2, P_{G_n,p_1}(B) \leq P_{G_n,p_2}(B) \text{ for any inctrasing event } B.$$

Theorem (C. 2010)

Let $\rho := \epsilon_2(p_c)$, where $\epsilon_2(\cdot)$ is as in previous theorem. Then $\rho > 0$. For any $p > p_c$ and $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}_p(\rho_n > \rho - \delta) = 1.$$